EFFECTIVE MODAL MASS & MODAL PARTICIPATION FACTORS
Revision K

By Tom Irvine
Email: tom@irvinemail.org

February 7, 2026

Introduction

The effective modal mass provides a quantitative measure of the *dynamic significance* of a vibration mode and its ability to be excited by a given loading mechanism.

For base-excited systems, modes with relatively high effective modal mass strongly participate in the imposed support motion and therefore dominate the dynamic response. Conversely, modes with low effective modal mass exhibit weak coupling to base motion and are difficult to excite through this mechanism.

For grounded systems subjected to applied forces, the same concept extends through the modal participation factor and the generalized modal force. In this case, a mode is considered dynamically significant if its mode shape has strong spatial correlation with the applied force distribution. Modes with large participation factors experience higher generalized forces and thus contribute more to the total response, even if their natural frequencies are high.

Modes with low effective modal mass or low participation factors have weak coupling to the applied forces and therefore contribute negligibly to the response, despite being present in the modal spectrum.

In practice, effective modal mass and participation factors together provide a rational basis for modal truncation, allowing engineers to:

- Identify which modes govern response for a given excitation type
- Justify neglecting higher-order or weakly coupled modes
- Ensure that retained modes capture the dominant inertia and force transmission paths

This distinction is especially important in structural dynamics, NVH, seismic analysis, and vibration qualification, where excitation mechanisms may involve a combination of base motion and externally applied forces.

NVH is automotive noise, vibration and harshness.

Number of Included Modes for Forced Response Analysis

Consider a modal transient or frequency response function analysis via the finite element method. Also consider that the system is a multi-degree-of-freedom system. For brevity, only a limited number of modes should be included in the analysis.

In structural dynamics, choosing the right number of modes is critical for balancing accuracy with computational efficiency. For a multi-degree-of-freedom (MDOF) system, the following guidelines are commonly used:

1. The 80%–90% Effective Mass Rule

The most standard engineering guideline is to include enough modes to account for at least 80% to 90% of the total cumulative effective mass in the direction of the excitation.

Why: Effective mass represents how much of the system's total mass is "participating" in a specific mode. If you capture 90% of the mass, you have likely captured the dominant inertial behavior of the structure.

Exception: For high-frequency localized excitations (like pyrotechnic shock), you may need even more.

2. The Frequency Range Rule (1.5x Rule)

Include all modes with natural frequencies up to 1.5 to 2 times the maximum frequency of interest in your excitation spectrum.

Example: If you are analyzing a system subject to vibration up to 2000 Hz, you should include all modes up to at least 3000 Hz (or even 4000 Hz).

Why: Even though higher-frequency modes are not being excited at their resonance, they contribute to the "static" or "residual" stiffness of the response.

3. Consider the Type of Excitation

Low-Frequency (Seismic/Wind): Usually, the first few fundamental modes (often <10) capture the vast majority of the response.

High-Frequency (Acoustics/Shock): You may need hundreds or thousands of modes to capture the localized, high-speed energy transfer.

Point Loads: If the load is applied at a single point, you may need more modes to accurately resolve the local deformation at that point compared to a distributed load (like gravity).

4. Convergence Study

When in doubt, perform a modal convergence study:

Run the analysis with an increasing number of modes. Compare the results (e.g., peak stress or displacement). If the change is negligible (typically <5%), your initial modal truncation was sufficient.

5. Using Residual Vectors (Static Correction)

If you cannot include enough modes to meet the 1.5x frequency rule, use Residual Vectors (sometimes called "Static Compensation" or "Mode Acceleration Method").

This technique mathematically accounts for the static contribution of the truncated higher-frequency modes without requiring the computational cost of solving for their eigenvalues. This is highly recommended in aerospace and automotive FEA.

Definitions

The equation definitions in this section are taken from Reference 1. Consider a discrete dynamic system governed by the following equation

$$\mathrm{M}\ddot{\bar{\mathrm{x}}} + \mathrm{K}\bar{\mathrm{x}} = \bar{\mathrm{F}} \tag{1}$$

where

- M is the mass matrix
- K is the stiffness matrix
- $\ddot{\bar{\mathrm{x}}}$ is the acceleration vector
- $\bar{\mathrm{x}}$ is the displacement vector
- $\bar{\mathrm{F}}$ is the forcing function or base excitation function

A solution to the homogeneous form of equation (1) can be found in terms of eigenvalues and eigenvectors. The eigenvectors represent vibration modes.

Let ϕ be the eigenvector matrix.

The system's generalized mass matrix $\widehat{m}$ is given by

$$\widehat{m} = \phi^T M \phi \quad (2)$$

Base Excitation Case

Let $\bar{r}$ be the influence vector which represents the displacements of the masses resulting from static application of a unit ground displacement. The influence vector induces a rigid body motion in all modes.

Define a coefficient vector $\bar{L}$ as

$$\bar{L} = \phi^T M \bar{r} \quad (3)$$

The modal participation factor matrix Γ_I for mode i is

$$\Gamma_i = \frac{L_i}{\widehat{m}_{ii}} \quad (4)$$

The effective modal mass $m_{eff,i}$ for mode i is

$$m_{eff,i} = \frac{L_i^2}{\widehat{m}_{ii}} \quad (5)$$

Note that $\widehat{m}_{ii} = 1$ for each index if the eigenvectors have been normalized with respect to the mass matrix.

Furthermore, the off-diagonal modal mass ($\widehat{m}_{ij}$, $i \neq j$) terms are zero regardless of the normalization and even if the physical mass matrix M has distributed mass. This is due to the orthogonality of the eigenvectors. The off-diagonal modal mass terms do not appear in equation (5), however. An example for a system with distributed mass is shown in Appendix F.

Example

Consider the two-degree-of-freedom system shown in Figure 1, with the parameters shown in Table 1.

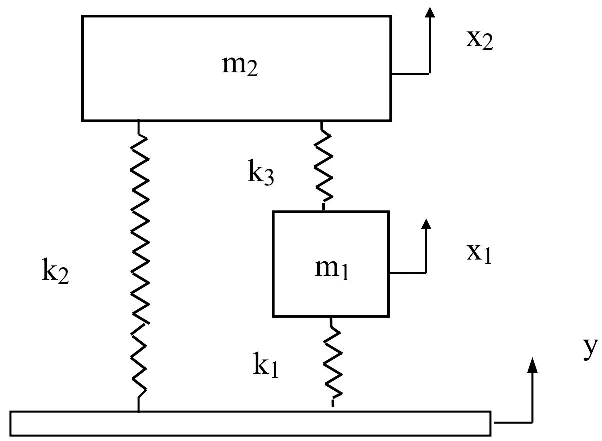

Figure 1.

Table 1. Parameters	
Variable	Value
m_1	2.0 kg
m_2	1.0 kg
k_1	1000 N/m
k_2	2000 N/m
k_3	3000 N/m

The homogeneous equation of motion is

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{6}$$

The mass matrix is

$$M = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \text{ kg} \tag{7}$$

The stiffness matrix is

$$K = \begin{bmatrix} 4000 & -3000 \\ -3000 & 5000 \end{bmatrix} \frac{\text{N}}{\text{m}} \tag{8}$$

The eigenvalues and eigenvectors can be found using the method in Reference 2. The eigenvalues are the roots of the following equation.

$$\det[K - \omega^2 M] = 0 \tag{9}$$

The eigenvalues are

$$\omega_1{}^2 = 901.9 \ \frac{\text{rad}}{\text{sec}^2} \tag{10}$$

$$\omega_1 = 30.03 \text{ rad/sec} \tag{11}$$

$$f_1 = 4.78 \text{ Hz} \tag{12}$$

$$\omega_2{}^2 = 6098 \ \frac{\text{rad}}{\text{sec}^2} \tag{13}$$

$$\omega_2 = 78.09 \text{ rad/sec} \tag{14}$$

$$f_2 = 12.4 \text{ Hz} \tag{15}$$

The eigenvector matrix is

$$\phi = \begin{bmatrix} 0.6280 & -0.3251 \\ 0.4597 & 0.8881 \end{bmatrix} \tag{16}$$

The eigenvectors were previously normalized so that the generalized mass is the identity matrix.

$$\widehat{m} = \phi^T M \phi \tag{17}$$

$$\widehat{m} = \begin{bmatrix} 0.6280 & 0.4597 \\ -0.3251 & 0.8881 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.6280 & -0.3251 \\ 0.4597 & 0.8881 \end{bmatrix} \tag{18}$$

$$\hat{m} = \begin{bmatrix} 0.6280 & 0.4597 \\ -0.3251 & 0.8881 \end{bmatrix} \begin{bmatrix} 1.2560 & -0.6502 \\ 0.4597 & 0.8881 \end{bmatrix} \tag{19}$$

$$\hat{m} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{20}$$

Again, $\bar{r}$ is the influence vector which represents the displacements of the masses resulting from static application of a unit ground displacement. For this example, each mass simply has the same static displacement as the ground displacement.

$$\bar{r} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{21}$$

The coefficient vector $\bar{L}$ is

$$\bar{L} = \phi^T M \bar{r} \tag{22}$$

$$\bar{L} = \begin{bmatrix} 0.6280 & 0.4597 \\ -0.3251 & 0.8881 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{23}$$

$$\bar{L} = \begin{bmatrix} 0.6280 & 0.4597 \\ -0.3251 & 0.8881 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \tag{24}$$

$$\bar{L} = \begin{bmatrix} 1.7157 \\ 0.2379 \end{bmatrix} \text{ kg} \tag{25}$$

The modal participation factor Γ_i for mode i is

$$\Gamma_i = \frac{L_i}{\hat{m}_{ii}} \tag{26}$$

The modal participation vector is thus

$$\Gamma = \begin{bmatrix} 1.7157 \\ 0.2379 \end{bmatrix} \tag{27}$$

The coefficient vector $\bar{L}$ and the modal participation vector Γ are identical in this example because the generalized mass matrix is the identity matrix.

The effective modal mass $m_{eff,i}$ for mode i is

$$m_{eff,i} = \frac{{L_i}^2}{\hat{m}_{ii}} \tag{28}$$

For mode 1,

$$m_{eff,1} = \frac{[1.7157 \text{ kg}]^2}{1\text{kg}} \tag{29}$$

$$m_{eff,1} = 2.944 \text{ kg} \tag{30}$$

For mode 2,

$$m_{eff,2} = \frac{[-0.2379 \text{ kg}]^2}{1\text{kg}} \tag{31}$$

$$m_{eff,2} = 0.056 \text{ kg} \tag{32}$$

Note that

$$m_{eff,1} + m_{eff,2} = 2.944 \text{ kg} + 0.056 \text{ kg} \tag{33}$$

$$m_{eff,1} + m_{eff,2} = 3 \text{ kg} \tag{34}$$

Thus, the sum of the effective masses equals the total system mass.

Also, note that the first mode has a much higher effective mass than the second mode.

Thus, the first mode can be readily excited by base excitation. On the other hand, the second mode is negligible in this sense.

From another viewpoint, the center of gravity of the first mode experiences a significant translation when the first mode is excited.

On the other hand, the center of gravity of the second mode remains nearly stationary when the second mode is excited.

Each degree-of-freedom in the previous example was a translation in the X-axis. This characteristic simplified the effective modal mass calculation.

In general, a system will have at least one translation degree-of-freedom in each of three orthogonal axes. Likewise, it will have at least one rotational degree-of-freedom about each of three orthogonal axes. The effective modal mass calculation for a general system is shown by the example in Appendix A. The example is from a real-world problem.

Applied Force Case

For a grounded system subjected to externally applied forces, the excitation is no longer represented by a ground-motion influence vector r, but instead by a load (input) distribution matrix B that maps applied forces into the physical degrees of freedom. The matrix B replaces the influence vector r used for base excitation.

The governing equation of motion may be written as

$$M\ddot{\bar{x}} + K\bar{x} = \bar{F} = B\,\bar{f}(t) \tag{35}$$

where

B is the input (or load) distribution matrix

$\bar{f}(t)$ is the vector of applied forces

The columns of B define the spatial distribution and direction of each applied force. For a single applied force, B reduces to a column vector. The participation factor for mode *i* and force input *j* is

$$\Gamma_{ij} = \phi_i^{\,T} B_j \tag{36}$$

where B_j is the *j*-th column of the load distribution matrix B.

This participation factor quantifies the degree to which mode *i* is excited by the spatial distribution of the applied force. Modes whose shapes are poorly correlated with the force distribution exhibit small participation factors and contribute minimally to the response.

The effective modal mass associated with mode *i* and force input *j* is defined as

$$m_{eff,ij} = \frac{\left(\phi_i{}^T B_j\right)^2}{\phi_i{}^T \, M \, \phi_i} \tag{37}$$

For mass normalized-eigenvectors,

$$m_{eff,ij} = \left(\phi_i{}^T B_j\right)^2 \tag{38}$$

For applied-force excitation, the quantity defined in equation (37) is sometimes referred to as an “effective modal mass” by analogy with base excitation. However, physically it represents a force-weighted modal inertia rather than a participating fraction of the total system mass.

Directionality for MDOF Systems

Effective modal mass is direction-dependent and excitation-dependent. A mode may have high effective mass in one direction and negligible effective mass in another.

References

1. M. Papadrakakis, N. Lagaros, V. Plevris; Optimum Design of Structures under Seismic Loading, European Congress on Computational Methods in Applied Sciences and Engineering, Barcelona, 2000.
2. T. Irvine, The Generalized Coordinate Method For Discrete Systems, Vibrationdata, 2000.
3. W. Thomson, Theory of Vibration with Applications 2nd Edition, Prentice Hall, New Jersey, 1981.
4. T. Irvine, Bending Frequencies of Beams, Rods, and Pipes, Rev M, Vibrationdata, 2010.
5. T. Irvine, Rod Response to Longitudinal Base Excitation, Steady-State and Transient, Rev B, Vibrationdata, 2009.
6. T. Irvine, Longitudinal Vibration of a Rod via the Finite Element Method, Revision B, Vibrationdata, 2008.

APPENDIX A

Equation of Motion, Isolated Avionics Component

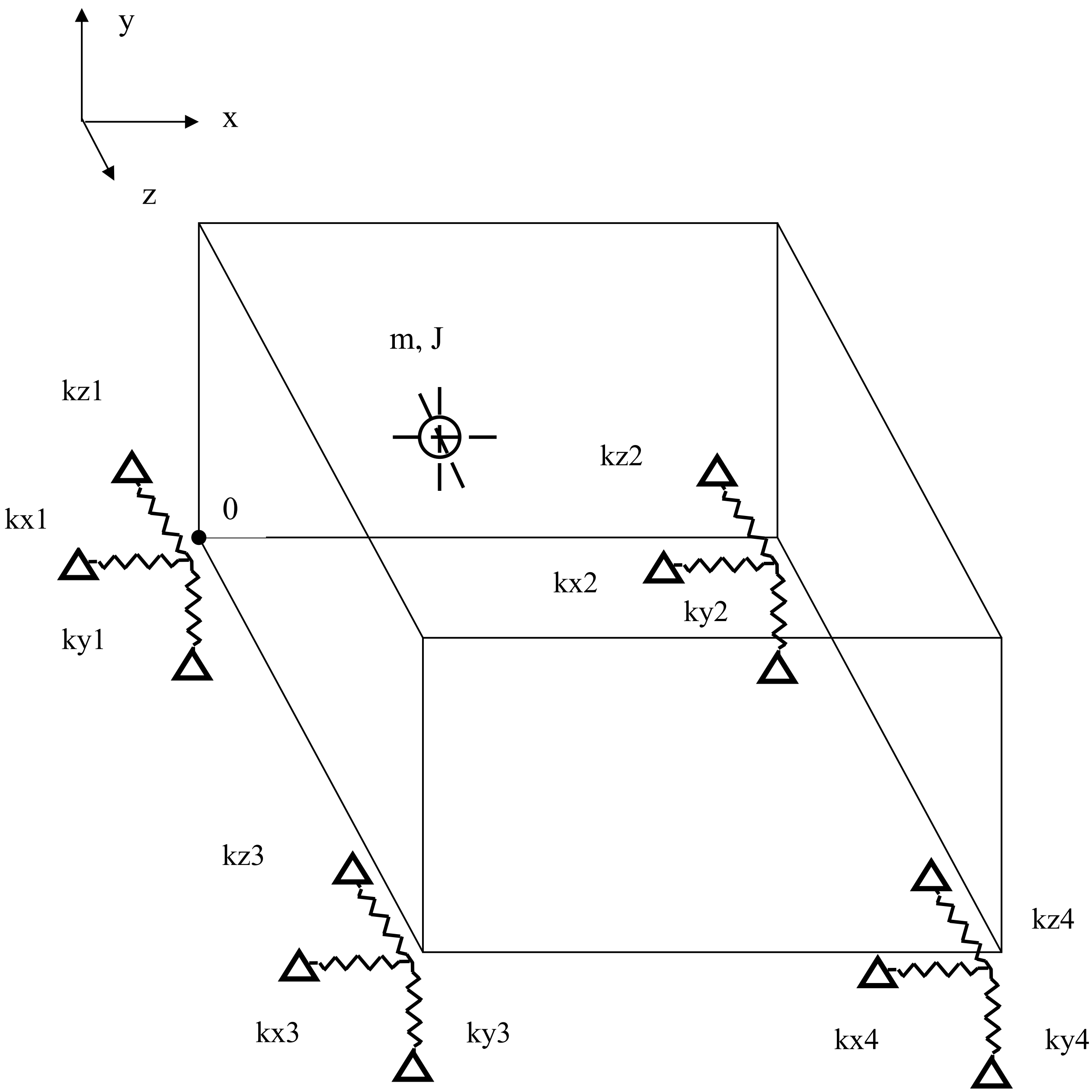

Figure A-1. Isolated Avionics Component Model

The mass and inertia are represented at a point with the circle symbol. Each isolator is modeled by three orthogonal DOF springs. The springs are mounted at each corner. The springs are shown with an offset from the corners for clarity. The triangles indicate fixed constraints. “0” indicates the origin.

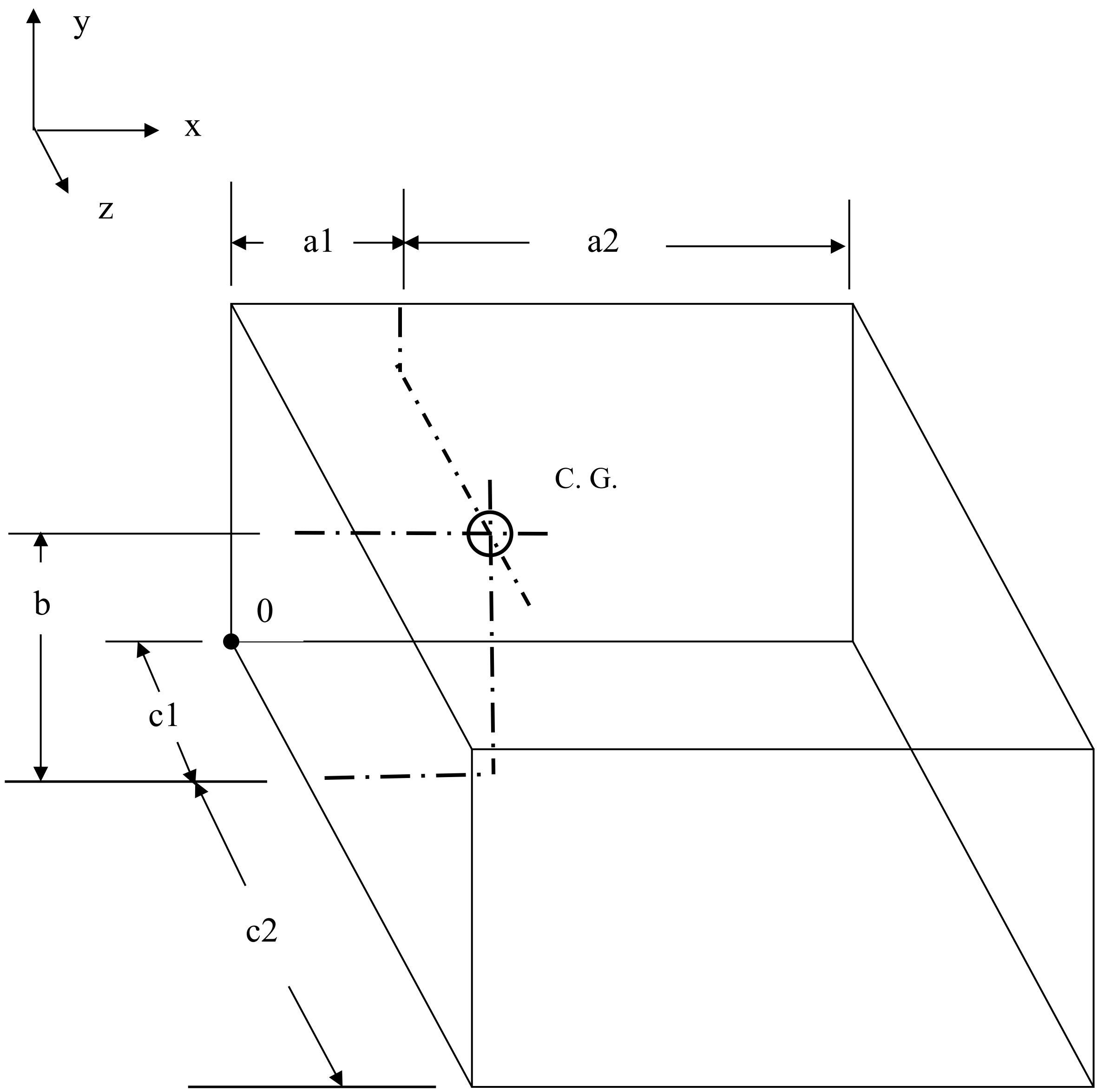

Figure A-2. Isolated Avionics Component Model with Dimensions

All dimensions are positive as long as the C.G. is “inside the box.” At least one dimension will be negative otherwise.

The equation of motion is

$$\underline{M}\begin{bmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\\ \ddot{\alpha}\\ \ddot{\beta}\\ \ddot{\theta}\end{bmatrix}+\underline{K}\begin{bmatrix}x\\ y\\ z\\ \alpha\\ \beta\\ \theta\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\\ 0\\ 0\\ 0\end{bmatrix}$$

(A-3)

The variables α, β and θ represent rotations about the X, Y, and Z axes, respectively.

Example

A mass is mounted to a surface with four isolators. The system has the following properties.

M	=	4.28 lbm
Jx	=	44.9 lbm in^2
Jy	=	39.9 lbm in^2
Jz	=	18.8 lbm in^2
kx	=	80 lbf/in
ky	=	80 lbf/in
kz	=	80 lbf/in
a1	=	6.18 in
a2	=	-2.68 in
b	=	3.85 in
c1	=	3. in
c2	=	3. in

The mass and stiffness matrices are shown in upper triangular form due to symmetry.

$$\underline{M} = \begin{bmatrix} m & 0 & 0 & 0 & 0 & 0 \\ & m & 0 & 0 & 0 & 0 \\ & & m & 0 & 0 & 0 \\ & & & J_x & 0 & 0 \\ & & & & J_y & 0 \\ & & & & & J_z \end{bmatrix} \tag{A-1}$$

$$\underline{K} = \begin{bmatrix} 4k_x & 0 & 0 & 0 & 2k_x(-c_1 + c_2) & 4k_x b \\ & 4k_y & 0 & 2k_y(c_1 - c_2) & 0 & 2k_y(-a_1 + a_2) \\ & & 4k_z & -4k_z b & 2k_z(a_1 - a_2) & 0 \\ & & & 4k_z b^2 + 2k_y({c_1}^2 + {c_2}^2) & 2k_z(-a_1 + a_2)b & k_y(-a_1 + a_2)(c_1 - c_2) \\ & & & & 2k_x({c_1}^2 + {c_2}^2) + 2k_z({a_1}^2 + {a_2}^2) & 2k_x(-c_1 + c_2)b \\ & & & & & 4k_x b^2 + 2k_y({a_1}^2 + {a_2}^2) \end{bmatrix} \tag{A-2}$$

Let B be the input (or load) distribution matrix. The B matrix for this example is the identity matrix provided that the C.G is the reference point.

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{(A-4)}$$

The participation factor for mode *i* and force input *j* is

$$\Gamma_{ij} = \phi_i{}^T B_j \quad \text{(A-5)}$$

where B_j is the *j*-th column of the load distribution matrix B.

For mass normalized-eigenvectors, the effective modal mass associated with mode *i* and force input *j* is defined as

$$m_{eff,ij} = \left(\phi_i{}^T B_j\right)^2 \quad \text{(A-6)}$$

The natural frequency results for the sample problem are calculated using the program: six_dof_iso.m.

The results are given in the next pages.

```
six_dof_iso.m   ver 1.2  March 31, 2005

 by Tom Irvine   Email: tomirvine@aol.com

 This program finds the eigenvalues and eigenvectors for a
 six-degree-of-freedom system.

 Refer to six_dof_isolated.pdf for a diagram.

 The equation of motion is:   M (d^2x/dt^2) + K x = 0

 Enter m (lbm)
 4.28
 Enter Jx (lbm in^2)
 44.9
 Enter Jy (lbm in^2)
 39.9
 Enter Jz (lbm in^2)
 18.8

 Note that the stiffness values are for individual springs

 Enter kx (lbf/in)
 80

 Enter ky (lbf/in)
 80

 Enter kz (lbf/in)
 80

 Enter a1 (in)
 6.18

 Enter a2 (in)
 -2.68

 Enter b (in)
 3.85

 Enter c1 (in)
 3


Enter c2 (in)
3


 The mass matrix is

m =

    0.0111         0         0         0         0         0
         0    0.0111         0         0         0         0
         0         0    0.0111         0         0         0
```

```
        0         0         0    0.1163         0         0
        0         0         0         0    0.1034         0
        0         0         0         0         0    0.0487


 The stiffness matrix is

k =

  1.0e+004 *

    0.0320         0         0         0         0    0.1232
         0    0.0320         0         0         0   -0.1418
         0         0    0.0320   -0.1232    0.1418         0
         0         0   -0.1232    0.7623   -0.5458         0
         0         0    0.1418   -0.5458    1.0140         0
    0.1232   -0.1418         0         0         0    1.2003


 Eigenvalues

lambda =

  1.0e+005 *

    0.0213    0.0570    0.2886    0.2980    1.5699    2.7318



  Natural Frequencies =
 1.      7.338 Hz
 2.      12.02 Hz
 3.      27.04 Hz
 4.      27.47 Hz
 5.      63.06 Hz
 6.      83.19 Hz

  Modes Shapes (rows represent modes)

            x            y            z        alpha         beta        theta
 1.      5.91        -6.81            0            0            0        -1.42
 2.         0            0         8.69        0.954       -0.744            0
 3.      7.17         6.23            0            0            0            0
 4.         0            0         1.04        -2.26        -1.95            0
 5.         0            0        -3.69         1.61         -2.3            0
 6.      1.96        -2.25            0            0            0          4.3

  Participation Factors (rows represent modes)

            x            y            z        alpha         beta        theta
 1.    0.0656      -0.0755            0            0            0      -0.0693
 2.         0            0       0.0963        0.111      -0.0769            0
 3.    0.0795       0.0691            0            0            0            0
 4.         0            0       0.0115       -0.263       -0.202            0
 5.         0            0      -0.0409        0.187       -0.238            0
 6.    0.0217       -0.025            0            0            0         0.21
```

Effective Modal Mass (rows represent modes)

	x	y	z	alpha	beta	theta
1.	0.0043	0.00569	0	0	0	0.0048
2.	0	0	0.00928	0.0123	0.00592	0
3.	0.00632	0.00477	0	0	0	0
4.	0	0	0.000133	0.069	0.0408	0
5.	0	0	0.00168	0.035	0.0566	0
6.	0.000471	0.000623	0	0	0	0.0439

Total Modal Mass

0.0111	0.0111	0.0111	0.116	0.103	0.0487

APPENDIX B

Modal Participation Factor for Applied Force

The following definition is taken from Reference 3. Note that the mode shape functions are unscaled. Hence, the participation factor is unscaled.

Consider a beam of length L loaded by a distributed force p(x,t).

Consider that the loading per unit length is separable in the form

$$p(x,t) = \frac{P_o}{L} p(x) f(t) \qquad \text{(B-1)}$$

The modal participation factor Γ_i for mode i is defined as

$$\Gamma_i = \frac{1}{L}\int_0^L p(x)\phi_i(x)dx \qquad \text{(B-2)}$$

where

$\phi_i(x)$ is the normal mode shape for mode i

APPENDIX C

Modal Participation Factor for a Beam

Let

$Y_n(x)$	=	mass-normalized eigenvectors
$m(x)$	=	mass per length

The participation factor is

$$\Gamma_n = \int_0^L m(x) Y_n(x) dx \qquad \text{(C-1)}$$

The effective modal mass is

$$m_{eff,n} = \frac{\left[\int_0^L m(x) Y_n(x) dx\right]^2}{\int_0^L m(x) [Y_n(x)]^2 dx} \qquad \text{(C-2)}$$

The eigenvectors should be normalized such that

$$\int_0^L m(x) [Y_n(x)]^2 dx = 1 \qquad \text{(C-3)}$$

Thus,

$$m_{eff,n} = [\Gamma_n]^2 = \left[\int_0^L m(x) Y_n(x) dx\right]^2 \qquad \text{(C-4)}$$

APPENDIX D

Effective Modal Mass Values for Bernoulli-Euler Beams

The results are calculated using formulas from Reference 4. The variables are

E	=	modulus of elasticity
I	=	area moment of inertia
L	=	length
ρ	=	(mass/length)

Table D-1. Bending Vibration, Beam Simply-Supported at Both Ends			
Mode	Natural Frequency ω_n	Participation Factor	Effective Modal Mass
1	$\frac{\pi^2}{L^2}\sqrt{EI/\rho}$	$\frac{2}{\pi}\sqrt{2\rho L}$	$\frac{8}{\pi^2}\rho L$
2	$4\frac{\pi^2}{L^2}\sqrt{EI/\rho}$	0	0
3	$9\frac{\pi^2}{L^2}\sqrt{EI/\rho}$	$\frac{2}{3\pi}\sqrt{2\rho L}$	$\frac{8}{9\pi^2}\rho L$
4	$16\frac{\pi^2}{L^2}\sqrt{EI/\rho}$	0	0
5	$25\frac{\pi^2}{L^2}\sqrt{EI/\rho}$	$\frac{2}{5\pi}\sqrt{2\rho L}$	$\frac{8}{25\pi^2}\rho L$
6	$36\frac{\pi^2}{L^2}\sqrt{EI/\rho}$	0	0
7	$49\frac{\pi^2}{L^2}\sqrt{EI/\rho}$	$\frac{2}{7\pi}\sqrt{2\rho L}$	$\frac{8}{49\pi^2}\rho L$

95% of the total mass is accounted for using the first seven modes.

Table D-2. Bending Vibration, Fixed-Free Beam			
Mode	Natural Frequency ω_n	Participation Factor	Effective Modal Mass
1	$\left[\frac{1.87510}{L}\right]^2 \sqrt{EI/\rho}$	$0.7830\sqrt{\rho L}$	$0.6131\rho L$
2	$\left[\frac{4.69409}{L}\right]^2 \sqrt{EI/\rho}$	$0.4339\sqrt{\rho L}$	$0.1883\rho L$
3	$\left[\frac{5\pi}{2L}\right]^2 \sqrt{EI/\rho}$	$0.2544\sqrt{\rho L}$	$0.06474\ \rho L$
4	$\left[\frac{7\pi}{2L}\right]^2 \sqrt{EI/\rho}$	$0.1818\sqrt{\rho L}$	$0.03306\ \rho L$

90% of the total mass is accounted for using the first four modes.

APPENDIX E

Rod, Longitudinal Vibration, Classical Solution

The results are taken from Reference 5.

Table E-1. Longitudinal Vibration of a Rod, Fixed-Free			
Mode	Natural Frequency ω_n	Participation Factor	Effective Modal Mass
1	0.5 π c / L	$\frac{2}{\pi}\sqrt{2\rho L}$	$\frac{8}{\pi^2}\rho L$
2	1.5 π c / L	$\frac{2}{3\pi}\sqrt{2\rho L}$	$\frac{8}{9\pi^2}\rho L$
3	2.5 π c / L	$\frac{2}{5\pi}\sqrt{2\rho L}$	$\frac{8}{25\pi^2}\rho L$

The longitudinal wave speed c is

$$c = \sqrt{E/\rho} \tag{E-1}$$

93% of the total mass is accounted for by using the first three modes.

APPENDIX F

This example shows a system with distributed or consistent mass matrix.

Rod, Longitudinal Vibration, Finite Element Method

Consider an aluminum rod with 1 inch diameter and 48 inch length. The rod has fixed-free boundary conditions.

A finite element model of the rod is shown in Figure F-1. It consists of four elements and five nodes. Each element has an equal length.

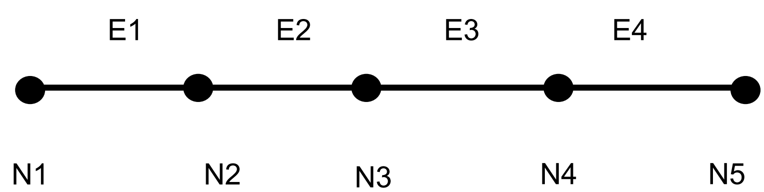

Figure F-1.

The boundary conditions are

$$U(0) = 0 \qquad \text{(Fixed end)} \qquad \text{(F-1)}$$

$$\left.\frac{dU}{dx}\right|_{x=L} = 0 \qquad \text{(Free end)} \qquad \text{(F-2)}$$

The natural frequencies and modes are determined using the finite element method in Reference 6.

The resulting eigenvalue problem for the constrained system has the following mass and stiffness matrices as calculated via Matlab script: rod_FEA.m.

Mass =

```
    0.0016    0.0004         0         0
    0.0004    0.0016    0.0004         0
         0    0.0004    0.0016    0.0004
         0         0    0.0004    0.0008
```

Stiffness =

```
  1.0e+006 *

    1.3090   -0.6545         0         0
   -0.6545    1.3090   -0.6545         0
         0   -0.6545    1.3090   -0.6545
         0         0   -0.6545    0.6545
```

The natural frequencies are

```
n         fn(Hz)
1         1029.9
2         3248.8
3         5901.6
4         8534.3
```

The mass-normalized eigenvectors in column format are

```
    5.5471   14.8349   18.0062   -9.1435
   10.2496   11.3542  -13.7813   16.8950
   13.3918   -6.1448   -7.4584  -22.0744
   14.4952  -16.0572   19.4897   23.8931
```

Let $\bar{r}$ be the influence vector which represents the displacements of the masses resulting from static application of a unit ground displacement.

The influence vector for the sample problem is

$$\bar{r} = [1 \quad 1 \quad 1 \quad 1]^T$$

The coefficient vector $\bar{L}$ is

$$\bar{L} = \phi^T M \bar{r} \tag{F-3}$$

where

ϕ^T	=	transposed eigenvector matrix
M	=	mass matrix

The coefficient vector for the sample problem is

$$\bar{L} = [0.0867 \quad 0.0233 \quad 0.0086 \quad -0.0021]^T \tag{F-3}$$

The modal participation factor matrix Γ_i for mode i is

$$\Gamma_i = L_i / \hat{m}_{ii} \tag{F-4}$$

Note that $\hat{m}_{ii}$= 1 for each index since the eigenvectors have been previously normalized with respect to the mass matrix.

Thus, for the sample problem,

$$\Gamma_i = L_i \tag{F-5}$$

The effective modal mass $m_{eff,i}$ for mode i is

$$m_{eff,i} = {L_i}^2 / \hat{m}_{ii} \tag{F-6}$$

Again, the eigenvectors are mass normalized.

Thus

$$m_{eff,i} = {L_i}^2 \qquad \text{(F-7)}$$

The effective modal mass for the sample problem is

$$m_{eff} = [0.0075 \quad 0.0005 \quad 0.0001 \quad 0.0000]^T$$

The model's total modal mass is 0.0081 lbf sec^2/in. This is equivalent to 3.14 lbm.

The true mass or the rod is 3.77 lbm.

Thus, the four-element model accounts for 83% of the true mass. This is a discretization error, not a limitation of modal mass theory.

This percentage can be increased by using a larger number of elements with corresponding shorter lengths.

APPENDIX G

Two-degree-of-freedom System, Static Coupling, Uniform Base Displacement

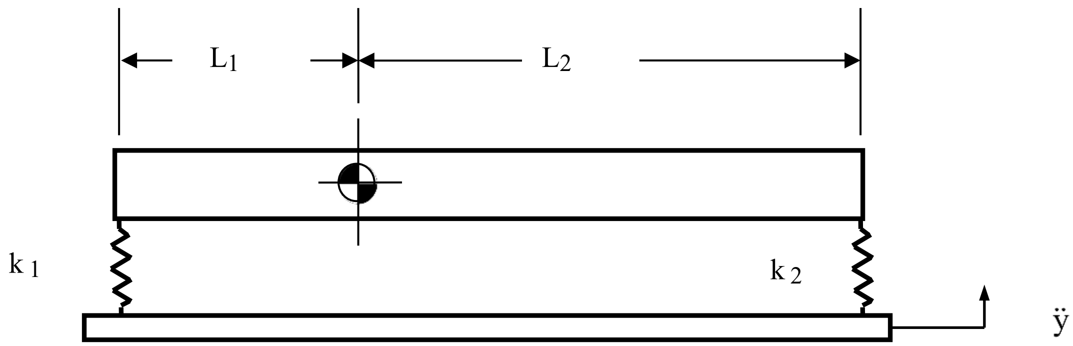

Figure G-1.

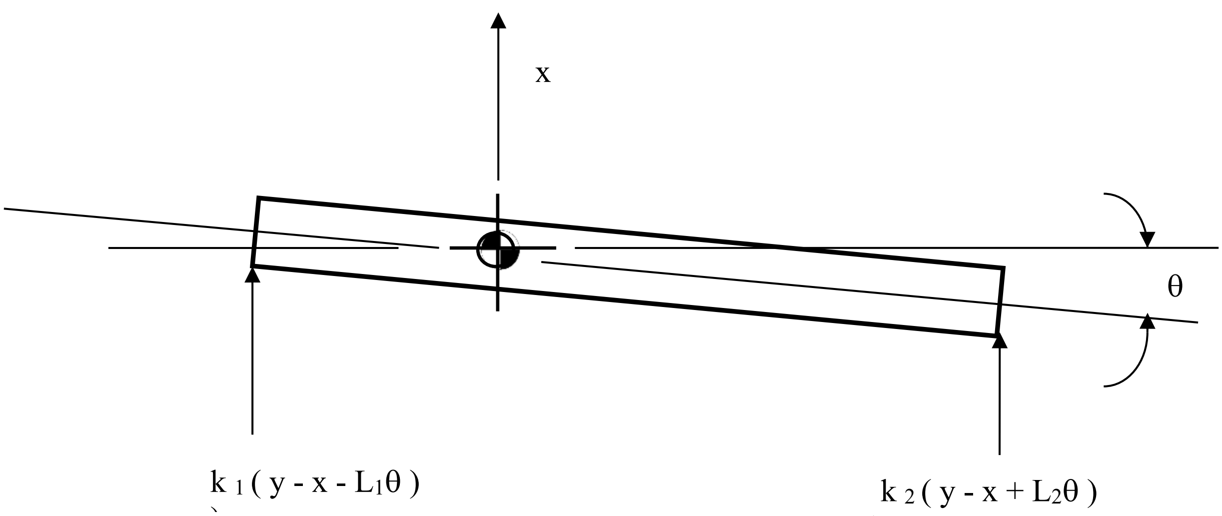

Figure G-2.

The free-body diagram is given in Figure G-2.

The system has a CG offset if $L_1 \neq L_2$.

The system is statically coupled if $k_1 L_1 \neq k_2 L_2$.

The rotation is positive in the clockwise direction.

The variables are

y	is the base displacement
x	is the translation of the CG
θ	is the rotation about the CG
m	is the mass
J	is the polar mass moment of inertia
k_i	is the stiffness for spring i
z_i	is the relative displacement for spring i

Sign Convention:

Translation: upward in vertical axis is positive.
Rotation: clockwise is positive.

Sum the forces in the vertical direction

$$\sum F = m\ddot{x} \quad \text{(G-1)}$$

$$m\ddot{x} = k_1(y - x - L_1\theta) + k_2(y - x + L_2\theta) \quad \text{(G-2)}$$

$$m\ddot{x} + k_1(-y + x + L_1\theta) + k_2(-y + x - L_2\theta) = 0 \quad \text{(G-3)}$$

$$m\ddot{x} - k_1 y + k_1 x + k_1 L_1\theta - k_2 y + k_2 x - k_2 L_2\theta = 0 \quad \text{(G-4)}$$

$$m\ddot{x} + (k_1 + k_2)x + (k_1 L_1 - k_2 L_2)\theta = (k_1 + k_2)y \quad \text{(G-5)}$$

Sum the moments about the center of mass.

$$\sum M = J\ddot{\theta} \quad \text{(G-6)}$$

$$J\ddot{\theta} = +k_1L_1(y - x - L_1\theta) - k_2L_2(y - x + L_2\theta) \qquad \text{(G-7)}$$

$$J\ddot{\theta} + k_1L_1(-y + x + L_1\theta) + k_2L_2(y - x + L_2\theta) = 0 \qquad \text{(G-8)}$$

$$J\ddot{\theta} - k_1y + k_1L_1x + k_1{L_1}^2\theta + k_2L_2y - k_2L_2x + k_2{L_2}^2\theta = 0 \qquad \text{(G-9)}$$

$$J\ddot{\theta} + [k_1L_1 - k_2L_2]x + \left[k_1{L_1}^2 + k_2{L_2}^2\right]\theta = (k_1L_1 - k_2L_2)y \qquad \text{(G-10)}$$

The equations of motion are

$$\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & k_1L_1 - k_2L_2 \\ k_1L_1 - k_2L_2 & k_1{L_1}^2 + k_2{L_2}^2 \end{bmatrix}\begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} k_1 + k_2 \\ k_1L_1 - k_2L_2 \end{bmatrix} y$$

(G-11)

The pseudo-static problem is

$$\begin{bmatrix} k_1 + k_2 & k_1L_1 - k_2L_2 \\ k_1L_1 - k_2L_2 & k_1{L_1}^2 + k_2{L_2}^2 \end{bmatrix}\begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} k_1 + k_2 \\ k_1L_1 - k_2L_2 \end{bmatrix} y \qquad \text{(G-12)}$$

Solve for the influence vector r by applying a unit displacement.

$$\begin{bmatrix} k_1 + k_2 & k_1L_1 - k_2L_2 \\ k_1L_1 - k_2L_2 & k_1{L_1}^2 + k_2{L_2}^2 \end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} k_1 + k_2 \\ k_1L_1 - k_2L_2 \end{bmatrix} \qquad \text{(G-13)}$$

$$\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{(G-14)}$$

Define a relative displacement z.

$$z = x - y \qquad \text{(G-15)}$$

$$x = z + y \qquad \text{(G-16)}$$

$$\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}\begin{bmatrix} \ddot{z} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} m\ddot{y} \\ 0 \end{bmatrix}$$

$$+ \begin{bmatrix} k_1 + k_2 & k_1 L_1 - k_2 L_2 \\ k_1 L_1 - k_2 L_2 & k_1 {L_1}^2 + k_2 {L_2}^2 \end{bmatrix}\begin{bmatrix} z \\ \theta \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & k_1 L_1 - k_2 L_2 \\ k_1 L_1 - k_2 L_2 & k_1 {L_1}^2 + k_2 {L_2}^2 \end{bmatrix}\begin{bmatrix} y \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} k_1 + k_2 \\ k_1 L_1 - k_2 L_2 \end{bmatrix}[y]$$

(G-17)

$$\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}\begin{bmatrix} \ddot{z} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & k_1 L_1 - k_2 L_2 \\ k_1 L_1 - k_2 L_2 & k_1 {L_1}^2 + k_2 {L_2}^2 \end{bmatrix}\begin{bmatrix} z \\ \theta \end{bmatrix} = - \begin{bmatrix} m\ddot{y} \\ 0 \end{bmatrix}$$

(G-18)

The equation is more formally

$$\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}\begin{bmatrix} \ddot{z} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & k_1 L_1 - k_2 L_2 \\ k_1 L_1 - k_2 L_2 & k_1 {L_1}^2 + k_2 {L_2}^2 \end{bmatrix}\begin{bmatrix} z \\ \theta \end{bmatrix} = - \begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \end{bmatrix}\ddot{y}$$

(G-19)

Solve for the eigenvalues and mass-normalized eigenvectors matrix ϕ using the homogeneous problem form of equation (G-9).

Define modal coordinates

$$\begin{bmatrix} z \\ \theta \end{bmatrix} = [\phi]\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \tag{G-20}$$

$$\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}[\phi]\begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & k_1 L_1 - k_2 L_2 \\ k_1 L_1 - k_2 L_2 & k_1 {L_1}^2 + k_2 {L_2}^2 \end{bmatrix}[\phi]\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = - \begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \end{bmatrix}\ddot{y}$$

(G-21)

Then premultiply by the transpose of the eigenvector matrix ϕ^T.

$$[\phi^T]\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}[\phi]\begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{bmatrix} + [\phi^T]\begin{bmatrix} k_1 + k_2 & k_1 L_1 - k_2 L_2 \\ k_1 L_1 - k_2 L_2 & k_1 {L_1}^2 + k_2 {L_2}^2 \end{bmatrix}[\phi]\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$$

$$= -[\phi^T]\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \end{bmatrix}\ddot{y}$$

(G-22)

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{bmatrix} + \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = -[\phi^T]\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \end{bmatrix}\ddot{y} \quad \text{(G-23)}$$

The participation factor vector is

$$\Gamma = \phi^T \begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \quad \text{(G-24)}$$

$$\Gamma = \phi^T \begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \phi^T \begin{bmatrix} m \\ 0 \end{bmatrix} \quad \text{(G-25)}$$

<u>Example</u>

Consider the system in Figure G-1. Assign the following values. The values are based on a slender rod, aluminum, diameter =1 inch, total length=24 inch.

Table G-1. Parameters	
Variable	Value
m	18.9 lbm
J	907 lbm in^2
k_1	20,000 lbf/in
k_2	20,000 lbf/in
L_1	8 in
L_2	16 in

The following parameters were calculated for the sample system via a Matlab script.

```
The mass matrix is

m =

    0.0490         0
         0    2.3497


 The stiffness matrix is

k =

       40000      160000
      160000     6400000


  Natural Frequencies =

     133.8 Hz
     267.9 Hz

  Modes Shapes (column format) =

         -4.4          1.029
       0.1486         0.6352

 Participation Factors =

    0.2156
    0.0504
```

```
 Effective Modal Mass

    0.0465
     0.0025
```

The total modal mass is 0.0490 lbf sec^2/in, equivalent to18.9 lbm.

APPENDIX H

<u>Two-degree-of-freedom System, Static & Dynamic Coupling, Uniform Base Excitation</u>

Repeat the example in Appendix G, but use the left end as the coordinate reference point.

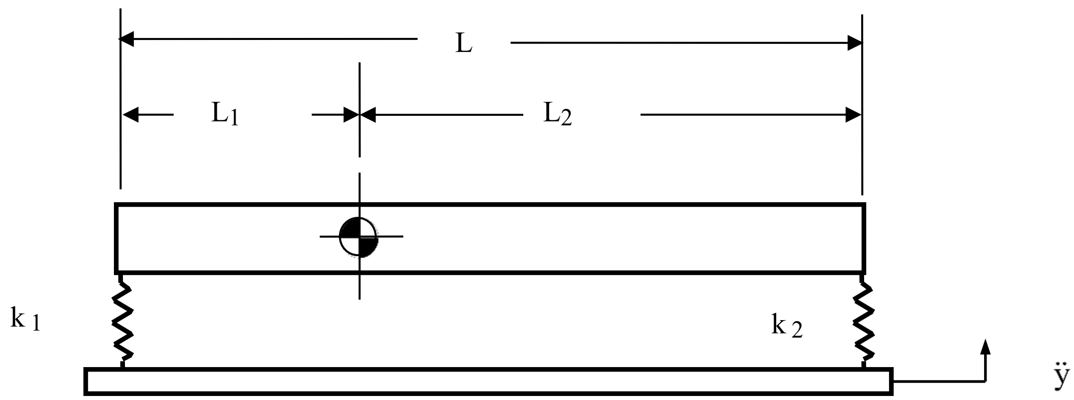

Figure H-1.

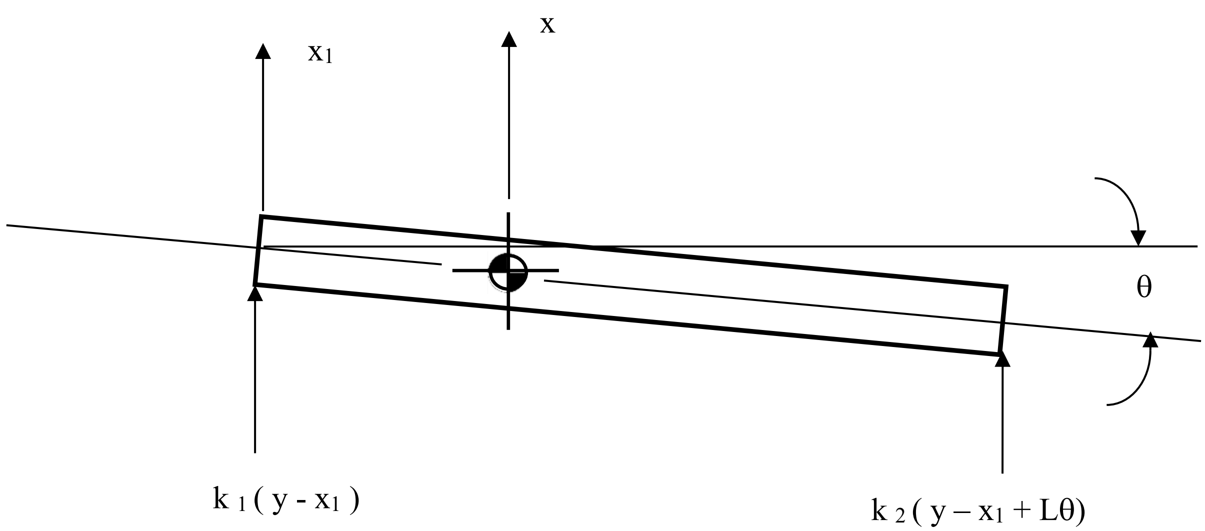

Figure H-2.

The free-body diagram is given in Figure H-2. Again, the displacement and rotation are referenced to the left end.

Sign Convention:

Translation: upward in vertical axis is positive.
Rotation: clockwise is positive.

Sum the forces in the vertical direction

$$\sum F = m\ddot{x} \tag{H-1}$$

$$m\ddot{x} = k_1(y - x_1) + k_2(y - x_1 + L\theta) \tag{H-2}$$

$$m\ddot{x} + k_1(-y + x_1) + k_2(-y + x_1 - L\theta) = 0 \tag{H-3}$$

$$m\ddot{x} - k_1 y + k_1 x_1 - k_2 y + k_2 x_1 - k_2 L\theta = 0 \tag{H-4}$$

$$m\ddot{x} + (k_1 + k_2)x_1 - k_2 L\theta = (k_1 + k_2)y \tag{H-5}$$

$$x = x_1 - L_1\theta \tag{H-6}$$

$$m(\ddot{x}_1 - L_1\ddot{\theta}) + (k_1 + k_2)x_1 - k_2 L\theta = (k_1 + k_2)y \tag{H-7}$$

$$m\ddot{x}_1 - mL_1\ddot{\theta} + (k_1 + k_2)x_1 - k_2 L\theta = (k_1 + k_2)y \tag{H-8}$$

Sum the moments about the left end.

$$\sum M_1 = J_1\ddot{\theta} \tag{H-9}$$

$$J_1\ddot{\theta} = -k_2 L\,(\,y - x_1 + L\theta\,) - mL_1(\ddot{x}\text{-}\ddot{x}_1) \tag{H-10}$$

$$J_1\ddot{\theta} + mL_1(\ddot{x} - \ddot{x}_1) + k_2 L\,(\,y - x_1 + L\theta\,) = 0 \tag{H-11}$$

$$J_1\ddot{\theta} + mL_1(\ddot{x} - \ddot{x}_1) + k_2 Ly - k_2 Lx_1 + k_2 L^2\theta = 0 \tag{H-12}$$

$$J_1\ddot{\theta} + mL_1(\ddot{x} - \ddot{x}_1) - k_2 Lx_1 + k_2 L^2\theta = -k_2 Ly \tag{H-13}$$

$$x = x_1 - L_1\theta \tag{H-14}$$

$$J_1\ddot{\theta} + mL_1(\ddot{x}_1 - L_1\ddot{\theta}\text{-}\ddot{x}_1) - k_2 Lx_1 + k_2 L^2\theta = -k_2 Ly \tag{H-15}$$

$$J_1\ddot{\theta} - mL_1\ddot{x}_1 - k_2Lx_1 + k_2L^2\theta = -k_2Ly \quad \text{(H-16)}$$

The equations of motion are

$$\begin{bmatrix} m & -mL_1 \\ -mL_1 & J_1 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2L \\ -k_2L & k_2L^2 \end{bmatrix}\begin{bmatrix} x_1 \\ \theta \end{bmatrix} = \begin{bmatrix} k_1 + k_2 \\ -k_2L \end{bmatrix} y \quad \text{(H-17)}$$

Note that

$$J_1 = J + mL_1^2 \quad \text{(H-18)}$$

$$\begin{bmatrix} m & -mL_1 \\ -mL_1 & J + mL_1^2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2L \\ -k_2L & k_2L^2 \end{bmatrix}\begin{bmatrix} x_1 \\ \theta \end{bmatrix} = \begin{bmatrix} k_1 + k_2 \\ -k_2L \end{bmatrix} y \quad \text{(H-19)}$$

The pseudo-static problem is

$$\begin{bmatrix} k_1 + k_2 & -k_2L \\ -k_2L & k_2L^2 \end{bmatrix}\begin{bmatrix} x_1 \\ \theta \end{bmatrix} = \begin{bmatrix} k_1 + k_2 \\ -k_2L \end{bmatrix} y \quad \text{(H-20)}$$

Solve for the influence vector r by applying a unit displacement.

$$\begin{bmatrix} k_1 + k_2 & -k_2L \\ -k_2L & k_2L^2 \end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} k_1 + k_2 \\ -k_2L \end{bmatrix} \quad \text{(H-21)}$$

$$\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{(H-22)}$$

The influence coefficient vector is the same as that in Appendix G.

The natural frequencies are obtained via a Matlab script. The results are:

```
Natural Frequencies
 No.      f(Hz)
1.        133.79
2.        267.93

  Modes Shapes (column format)

ModeShapes =

    5.5889    4.0527
    0.1486    0.6352


 Participation Factors =

    0.2155
  -0.05039

 Effective Modal Mass =

   0.04642
  0.002539
```

The total modal mass is 0.0490 lbf sec^2/in, equivalent to18.9 lbm.